

# High order cumulants of the azimuthal anisotropy in the dilute-dense limit: Connected graphs.

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We analytically compute higher order cumulants of the azimuthal anisotropy,  $c_2\{2m\}$ , and corresponding  $v_2\{2m\}$  at high transverse momentum in the dilute-dense limit. The dense target is considered in the framework of the McLerran-Venugopalan model. The absolute values of the harmonics  $v_2\{2m\}$  of the azimuthal anisotropy are approximately equal,  $|v_2\{2m\}| \approx |v_2\{2m'\}|$ , for large  $m$  and  $m'$ . However, the harmonics with order  $2m = 4n$  are complex. We argue that this is a generic property of connected graphs, which remains true in the dense-dense limit.

## I. INTRODUCTION

Large azimuthal asymmetries observed in p+Pb collisions at the LHC [1–3] and in d+Au collisions at RHIC [4] have usually been described by hydrodynamics [5] or the “glasma graph” [6] computed in the framework of the Color Glass Condensate (CGC). Both approaches resulted in a fairly good description of the data for two particle correlations. The origin of the asymmetries is, however, very different: in hydrodynamics the asymmetry is related to the azimuthal anisotropy for a single particle, while in the Color Glass Condensate <sup>1</sup> it is related to the correlation of gluons in the dense target (and the dense projectile in case of the dense-dense limit).

Higher order cumulants of azimuthal anisotropy and associated harmonics are a very sensitive probe of *collectivity* in the system because they enhance non-trivial  $m$ -particle correlations. These correlations naturally appear in a hydrodynamical description of high-energy p-A and A-A collisions. However, we warn the reader from equating hydrodynamics and collectivity, because intrinsic correlations between  $m$ -particle and apparent collectivity may arise owing to many-particle dynamics of partons in the target wave function. As an example, we note that the gluon saturation is already a genuinely collective/many-particle phenomena.

Nonetheless, we show herein that higher order cumulants are capable of narrowing down the list of models used to describe high-energy hadron collisions.

In this article we consider the dilute-dense limit and compute higher order cumulants of the azimuthal anisotropy. We show that fully connected graphs, which are topologically equivalent to the glasma graph of the Color Glass Condensate, produce positive four-particle cumulants, i.e. *complex*  $v_2\{4\}$ . We also show that the *absolute* value of the harmonics  $v_2\{m\}$  for large  $m$  is non-zero and approximately independent of  $m$ .

## II. CALCULATIONS OF HIGH ORDER CUMULANTS OF AZIMUTHAL ANISOTROPY

### A. S-matrix

In the dilute-dense limit, the projectile is modeled as a collection of partons scattering off the classical field of the target. We treat the target in the semi-classical approximation following the McLerran-Venugopalan model. Scattering of a parton off the target is quantified by the S-matrix

$$\mathcal{S}_1(\mathbf{r}, \mathbf{b}) \equiv \frac{1}{N_c} \text{tr} V^\dagger(\mathbf{x}) V(\mathbf{y}), \quad (1)$$

where the dipole radius and the impact parameter are  $\mathbf{r} \equiv \mathbf{x} - \mathbf{y}$  and  $\mathbf{b} \equiv \frac{1}{2}(\mathbf{x} + \mathbf{y})$  respectively.  $V(\mathbf{x})$  is the Wilson line describing propagation of the parton in the field of the target

$$V(\mathbf{x}) = \mathbb{P} \exp \left\{ ig \int dx^- A^+(x^-, \mathbf{x}) \right\}. \quad (2)$$

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<sup>1</sup> We note that the “glasma graph” does not exhaust all the sources of the azimuthal anisotropy within the CGC framework, see e.g. Refs. [7–11].

We restrict our consideration to the region of high transverse momentum, where we can perform expansion of the scattering cross section with respect to the dipole size  $r$ , the variable conjugate to the transverse momentum. We perform explicit derivation only for quarks scattering off the target, or for the fundamental representation of the Wilson line, Eq. (2). The final result for the cumulants of an even order <sup>2</sup>, however, is true for gluons as well. This arises from the fact that representation-dependent Casimir factors cancel out in normalized observables as cumulants.

Performing the gradient expansion for the vector potential

$$A^+(x^-, \mathbf{b} \pm \frac{\mathbf{r}}{2}) \approx A^+(x^-, \mathbf{b}) \pm \frac{\mathbf{r}}{2} \cdot \nabla A^+(x^-, \mathbf{b}) \quad (3)$$

we get to the lowest order in  $|\mathbf{r}|$

$$\mathcal{S}_1(\mathbf{r}, \mathbf{b}) - 1 = \frac{(ig)^2}{2N_c} \text{tr} (\mathbf{r} \cdot \mathbf{E}(\mathbf{b}))^2 + \mathcal{O}(r^3), \quad (4)$$

where the electric field of the target reads

$$E_i(\mathbf{b}) = -\partial_i \left( \int dx^- A^+(x^-, \mathbf{b}) \right). \quad (5)$$

Analogously, the  $m$ -quark S-matrix is given by

$$\mathcal{S}_m(\mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_m, \mathbf{b}_m) - 1 = \left( \frac{(ig)^2}{2N_c} \right)^m \prod_{i=1}^m \text{tr} (\mathbf{r}_i \cdot \mathbf{E}(\mathbf{b}_i))^2. \quad (6)$$

In order to proceed with the computations we need to specify the field-field correlator, which we adopt from the MV model [12]

$$\frac{g^2}{N_c} \langle E_i^a(\mathbf{b}_1) E_j^b(\mathbf{b}_2) \rangle = \frac{1}{N_c^2 - 1} \delta^{ab} \delta_{ij} Q_s^2 \Delta(\mathbf{b}_1 - \mathbf{b}_2), \quad (7)$$

where a general form of the impact parameter dependence of the correlator  $\Delta(\mathbf{b})$  with the Fourier image  $\tilde{\Delta}(\mathbf{k})$  was assumed.  $\Delta(\mathbf{b})$  is normalized such that  $\Delta(0) = 1$ .

## B. Cumulants of azimuthal anisotropy

The derivation of the cumulants of azimuthal anisotropy reduce to the computation of the azimuthal dependence of the fully connected expectation value of the S-matrix for  $m$  particles. The  $m$ -th order cumulant is given by

$$c_2\{m = 2n\} = \langle \exp[i2(\phi_1 + \phi_2 + \dots + \phi_n - \phi_{n+1} - \phi_{n+2} - \dots - \phi_{2n})] \rangle_\phi, \quad (8)$$

where the average with respect to the angular coordinates is defined by

$$\langle f(\phi_1, \dots, \phi_m) \rangle_\phi = \frac{\int d\phi_1 \dots d\phi_m f(\phi_1, \dots, \phi_m) \langle S_m(\mathbf{r}_1, \dots, \mathbf{r}_m) - 1 \rangle^{\text{conn.}}}{\int d\phi_1 \dots d\phi_m \langle S_m(\mathbf{r}_1, \dots, \mathbf{r}_m) - 1 \rangle}. \quad (9)$$

The azimuthal angles  $\phi_m$  in the laboratory frame characterize each particle (in our case, a dipole with  $\mathbf{r}_m = (r_m \cos \phi_m, r_m \sin \phi_m)$ ).

To simplify the notation we introduced the S-matrix averaged with respect to the impact parameters, as follows

$$S_m(\mathbf{r}_1, \dots, \mathbf{r}_m) = \frac{1}{S_\perp^m} \int d^2b_1 \dots d^2b_m S_m(\mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_m, \mathbf{b}_m), \quad (10)$$

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<sup>2</sup> For the odd orders one has to be a bit more careful, since the adjoint representation of SU(3) is real and thus cannot give rise to odd cumulants. For the fundamental representation this is, in principle, possible; however, owing to the approximate equality between the number of quarks and antiquarks in the projectile, we believe that odd cumulants are of a negligible magnitude.

where  $S_\perp$  is the transverse area of the projectile. In Eq. (10)  $\langle S_m(\mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_m, \mathbf{b}_m) - 1 \rangle^{\text{conn.}}$  denotes the fully connected contribution to the S-matrix. We perform the angular average in the  $r$ -space, because to this order of the S-matrix expansion it is equivalent to the angular average in the momentum space.

The denominator in Eq. (36) is dominated by the disconnected graph (see Fig. 1) with the corrections suppressed by powers of  $1/N_c$ :

$$\langle S_m(\mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_m, \mathbf{b}_m) - 1 \rangle = \left( \frac{(ig)^2}{2N_c} \right)^m \prod_{i=1}^m \langle \text{tr}(\mathbf{r}_i \cdot \mathbf{E}(\mathbf{b}_i))^2 \rangle + \mathcal{O}(N_c^{-2}). \quad (11)$$

Thus to the leading order in  $N_c$ , the denominator

$$\langle S_m(\mathbf{r}_1, \dots, \mathbf{r}_m) - 1 \rangle \approx \left( -\frac{Q_s^2}{4} \right)^m \prod_{i=1}^m r_i^2. \quad (12)$$

The numerator in Eq. (36) involves all possible contractions that generate the fully connected graphs. There are  $(2m-2)!!$  ways to contract  $S_m(\mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_m, \mathbf{b}_m)$  in a fully connected way. Here we show only one term, the rest of  $(2m-2)!! - 1$  terms can be obtained by permutations:

$$\begin{aligned} \langle S_m(\mathbf{r}_1, \mathbf{b}_1, \dots, \mathbf{r}_m, \mathbf{b}_m) - 1 \rangle^{\text{conn.}} &= \left( \frac{-Q_s^2}{4} \right)^m \frac{1}{(N_c^2 - 1)^{m-1}} \Delta(\mathbf{b}_1 - \mathbf{b}_2) \Delta(\mathbf{b}_2 - \mathbf{b}_1) \cdots \Delta(\mathbf{b}_{m-1} - \mathbf{b}_m) \Delta(\mathbf{b}_m - \mathbf{b}_1) \\ &\quad (\mathbf{r}_1 \mathbf{r}_2)(\mathbf{r}_2 \mathbf{r}_3) \cdots (\mathbf{r}_{m-1} \mathbf{r}_m)(\mathbf{r}_m \mathbf{r}_1) + \text{permutations}. \end{aligned} \quad (13)$$

Averaging with respect to the impact parameter can be best done using Fourier transformation:

$$\frac{1}{S_\perp^m} \int d^2 b_1 \cdots d^2 b_m \Delta(\mathbf{b}_1 - \mathbf{b}_2) \Delta(\mathbf{b}_2 - \mathbf{b}_1) \cdots \Delta(\mathbf{b}_{m-1} - \mathbf{b}_m) \Delta(\mathbf{b}_m - \mathbf{b}_1) = \frac{1}{S_\perp^{m-1}} \int \frac{d^2 k}{(2\pi)^2} \tilde{\Delta}^m(k). \quad (14)$$

In what follows we consider two correlation functions: a Gaussian

$$\Delta_G(\mathbf{b}) = \exp \left( -\frac{\mathbf{b}^2}{\sigma^2} \right) \quad (15)$$

and an exponential

$$\Delta_E(\mathbf{b}) = \exp \left( -\sqrt{2} \frac{b}{\sigma} \right). \quad (16)$$

Both functions are introduced such that

$$\frac{1}{S_\perp} \int d^2 b \Delta_{G,E}(\mathbf{b}) = \frac{1}{S_\perp} \pi \sigma^2 = \frac{S_\perp^c}{S_\perp} = \xi. \quad (17)$$

Here  $\xi$  is the ratio of the correlated area,  $S_\perp^c$ , to the area of the projectile,  $S_\perp$  (the proton in p-A collisions).

For these two cases we have

$$\frac{1}{S_\perp^m} \int d^2 b_1 \cdots d^2 b_m \Delta_G(\mathbf{b}_1 - \mathbf{b}_2) \Delta_G(\mathbf{b}_2 - \mathbf{b}_1) \cdots \Delta_G(\mathbf{b}_{m-1} - \mathbf{b}_m) \Delta_G(\mathbf{b}_m - \mathbf{b}_1) = \frac{\xi^{m-1}}{m}, \quad (18)$$

$$\frac{1}{S_\perp^m} \int d^2 b_1 \cdots d^2 b_m \Delta_E(\mathbf{b}_1 - \mathbf{b}_2) \Delta_E(\mathbf{b}_2 - \mathbf{b}_1) \cdots \Delta_E(\mathbf{b}_{m-1} - \mathbf{b}_m) \Delta_E(\mathbf{b}_m - \mathbf{b}_1) = \frac{\xi^{m-1}}{3m-2}. \quad (19)$$

Thus after averaging with respect to the impact parameter we obtain

$$\begin{aligned} \langle S_m(\mathbf{r}_1, \dots, \mathbf{r}_m) - 1 \rangle^{\text{conn.}} &= \left( \frac{-Q_s^2}{4} \right)^m \left( \frac{\xi}{N_c^2 - 1} \right)^{m-1} \left\{ \frac{m^{-1}}{(3m-2)^{-1}} \right\} (\mathbf{r}_1 \mathbf{r}_2)(\mathbf{r}_2 \mathbf{r}_3) \cdots (\mathbf{r}_{m-1} \mathbf{r}_m)(\mathbf{r}_m \mathbf{r}_1) \\ &\quad + \text{permutations}. \end{aligned} \quad (20)$$

Here the upper (lower) line corresponds to the Gaussian (exponential) correlator. This result can be used to compute the factorial moments along similar lines as for the dense-dense limit, see Ref. [13]<sup>3</sup>.

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<sup>3</sup> Indeed a straightforward computation of the factorial moments gives  $m_q = (q-1)! \frac{K}{2} \left( \frac{\bar{n}}{K} \right)^q \left\{ \frac{q^{-1}}{(3q-2)^{-1}} \right\}$ , where  $K = (N_c^2 - 1)/\xi$  and  $\bar{n}$  is the average number of scattered partons. Note that this agrees only logarithmically with the usual negative binomial result:  $m_q^{\text{NBD}} = (q-1)! K \left( \frac{\bar{n}}{K} \right)^q$ , which was obtained in the dense-dense limit in Ref. [13].

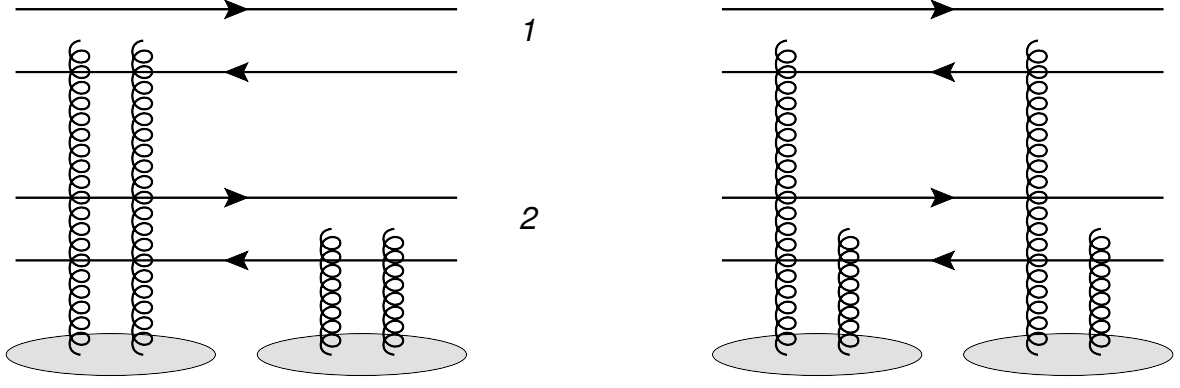


FIG. 1: The disconnected (left) and connected (right) contributions to the S-matrix for 2 particles.

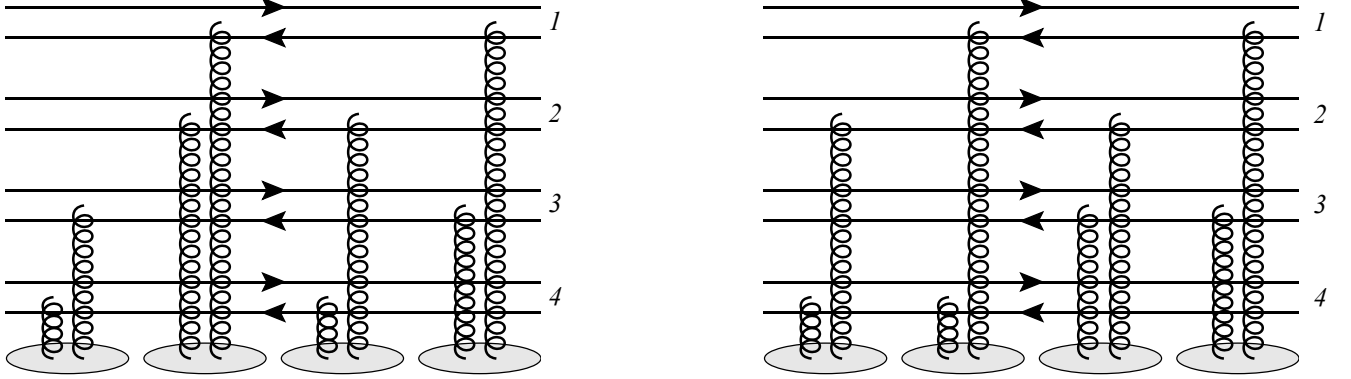


FIG. 2: An example of two connected contributions to the S-matrix for four particles. After performing angular average of  $\exp\{2i(\phi_1 + \phi_2 - \phi_3 - \phi_4)\}$  only the right diagram provides a non-zero contribution: it involves contractions of dipoles 1 or 2 with 3 or 4. The left (right) diagram is proportional to  $(\mathbf{r}_1\mathbf{r}_3)(\mathbf{r}_2\mathbf{r}_4)(\mathbf{r}_1\mathbf{r}_2)(\mathbf{r}_3\mathbf{r}_4) \left( (\mathbf{r}_1\mathbf{r}_3)(\mathbf{r}_1\mathbf{r}_4)(\mathbf{r}_2\mathbf{r}_3)(\mathbf{r}_2\mathbf{r}_4) \right)$ .

The next step is the integration with respect to all  $\phi_i$  of  $e^{2i(\phi_1 + \phi_2 + \dots + \phi_n - \phi_{n+1} - \phi_{n+2} - \dots - \phi_{2n})}$  weighted with  $S_m$ , where  $2n = m$ . Not all the terms in Eq. (20) contribute to this average. In fact, the term we wrote explicitly down vanishes after the integration. However, the  $m!!(m-2)!!$  nonzero terms remain<sup>4</sup>. Those are defined by all possible contractions of the terms entering with opposite signs before  $\phi$ 's in  $e^{2i(\phi_1 + \phi_2 + \dots + \phi_n - \phi_{n+1} - \phi_{n+2} - \dots - \phi_{2n})}$ . An example of a term resulting in a nonzero contribution is  $\propto (\mathbf{r}_1\mathbf{r}_{n+1})(\mathbf{r}_1\mathbf{r}_{n+2})(\mathbf{r}_2\mathbf{r}_{n+2})(\mathbf{r}_2\mathbf{r}_{n+3}) \dots (\mathbf{r}_{n-1}\mathbf{r}_{2n-1})(\mathbf{r}_{n-1}\mathbf{r}_{2n})(\mathbf{r}_n\mathbf{r}_{2n})(\mathbf{r}_n\mathbf{r}_{n+1})$ . See also an explicit example in Fig. 2. The angular integration of every such term gives an extra factor of  $1/2$ . Hence we obtain<sup>5</sup>

$$c_2^G\{m\} = \frac{m!!(m-2)!!}{m 2^m} \left( \frac{\xi}{N_c^2 - 1} \right)^{m-1}, \quad (21)$$

$$c_2^E\{m\} = \frac{m!!(m-2)!!}{(3m-2)2^m} \left( \frac{\xi}{N_c^2 - 1} \right)^{m-1} \quad (22)$$

for the Gaussian and exponential correlators respectively. The harmonics of the azimuthal anisotropy are related to the cumulants by

$$v_2^m\{m\} = \kappa_m c_2\{m\}, \quad (23)$$

<sup>4</sup> This is in a quantitative contrast to computations of the factorial moments, see Ref. [13] for the dense-dense limit computation of the factorial moments.

<sup>5</sup> This result can be further simplified by taking into account that for even  $m$ :  $m!! = 2^{m/2}(m/2)!!$ .

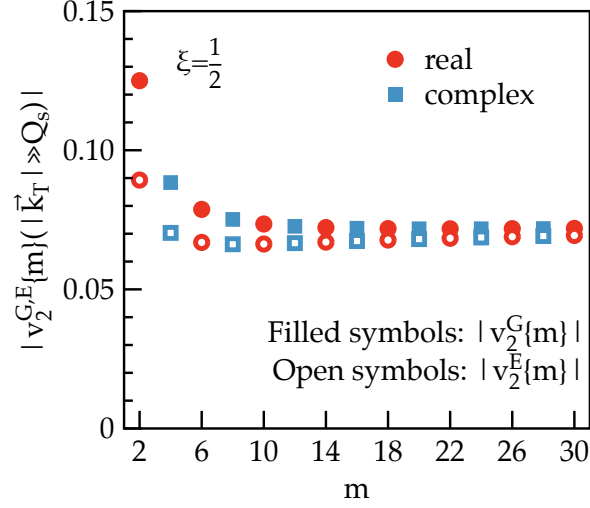


FIG. 3: The absolute value of  $v_2^{G,E}\{m\}$  as a function of  $m$ . The filled (open) symbols correspond to  $|v_2^G\{m\}|$  ( $|v_2^E\{m\}|$ ). The filled and open circles (squares) denote real (complex)  $v_2\{m\}$ . The calculations are performed for  $\xi = 1/2$ . Phenomenologically relevant values for  $\xi$  can be found elsewhere [9].

where, as shown in Appendix,

$$\kappa_{2n} = (-1)^{n+1} \left[ n!(n-1)! \sum_{k=1}^{\infty} \left( \frac{2}{j_{0,k}} \right)^{2n} \right]^{-1}. \quad (24)$$

Here  $j_{0,k}$  is the  $k$ -th zero of Bessel function  $J_0(x)$ . For a large orders  $n$  we have

$$\kappa_{2n} \approx \frac{(-1)^{n+1}}{n!(n-1)!} \left( \frac{j_{0,1}}{2} \right)^{2n}. \quad (25)$$

Finally we obtain for even  $m$

$$(v_2^G\{m\})^m = \frac{(-1)^{\frac{m}{2}+1}}{2m \sum_{k=1}^{\infty} \left( \frac{2}{j_{0,k}} \right)^m} \left( \frac{\xi}{N_c^2 - 1} \right)^{m-1} \quad \text{and} \quad (v_2^E\{m\})^m = \frac{(-1)^{\frac{m}{2}+1}}{2(3m-2) \sum_{k=1}^{\infty} \left( \frac{2}{j_{0,k}} \right)^m} \left( \frac{\xi}{N_c^2 - 1} \right)^{m-1}. \quad (26)$$

This result also holds for gluons scattering off the target.

For a large order  $m$  we get for the absolute value of the harmonics

$$\lim_{m \rightarrow \infty} |v_2\{m\}| = \frac{\xi}{N_c^2 - 1} \frac{j_{0,1}}{2} \quad (27)$$

for both  $c_2^G\{m\}$  and  $c_2^E\{m\}$ . Although we were unable to prove this rigorously, we believe that this result holds for any short range  $\Delta(\mathbf{b})$ .

We established the equality of the absolute values of high order harmonics. The main point of this article is that the fully connected graphs give *positive* cumulants of *any* order and since  $\kappa_m$  alternate sign depending on the order  $m$ , every second  $v_2\{m\}$  is *complex*! The lowest order at which  $v_2\{m\}$  becomes complex is the *fourth*

$$(v_2^{G,E}\{4\})^4 = -c_2\{4\} = - \left\{ \frac{1/4}{1/10} \right\} \left( \frac{\xi}{N_c^2 - 1} \right)^3 < 0, \quad (28)$$

as is also illustrated in Fig. 3. One can extend this consideration for the high momentum region of the dense-dense limit, the so-called “glasma” graph, and show that in this case  $v_2\{4\}$  is complex as well.

### III. DISCUSSION

As we established in the previous section, while the absolute values of high order harmonics are approximately equal to each other and therefore follow the hierarchy observed in high-energy p-A collisions ( $v_2\{4\} \approx v_2\{6\} \approx v_2\{8\}$ ), every second coefficient starting from  $v_2\{4\}$  is complex in contradiction to what is seen in experiments at high multiplicities. For small multiplicities, however, experimental  $c_2\{4\}$  is positive, i.e.  $v_2\{4\}$  is complex. This implies that the connected graphs considered here may potentially describe only low multiplicity collisions. High multiplicities are dominated by other mechanisms. This might include final state interactions (i.e. rescattering, not related to the initial state CGC dynamics) or an initial state effect not accounted for by the formalism considered above. The former possibility is discussed in detail in the literature, see Refs. [5, 14]. The possibility for the latter was put forward by Kovner and Lublinsky in Ref. [7] and recently developed in Refs. [9–11, 15]: the main idea is that the target’s electric field being a vector necessarily should points to some direction of the transverse impact parameter space and form a domain structure of the target. Partons scattering off this electric field receive the same momentum kick and this generates multi-particle long-range rapidity correlation. We stress that this correlation arises from a single particle azimuthal anisotropy and thus the higher order harmonics are real if the contribution, defined by the disconnected graphs, dominates over the connected graphs (intrinsic correlations) considered in this paper. An interested reader is referred to Ref. [10] for the detailed analysis of the fourth order cumulant with connected and disconnected graphs taken into account. Recent MV model calculations and JIMWLK evolution [11] indeed showed that the mentioned one-particle azimuthal anisotropy is present in the S-matrix and it does not vanish at small  $x$ .

Besides the above, the results of this manuscript were derived in the framework of the MV model for the dense target; that is we explicitly assumed Gaussian correlations, see Eq. (7). As was shown in Ref. [17], JIMWLK evolution generates higher order correlations, e.g. a four point function  $\langle E^a E^b E^c E^d \rangle$  will involve terms of the form  $\frac{1}{N_c} f^{abe} f^{cde}$ . These corrections are not necessarily large as shown in Ref. [18] by numerical simulations for various higher order function.

### IV. APPENDIX: FROM CUMULANTS TO HARMONICS OF THE AZIMUTHAL ANISOTROPY

In this Appendix, we derive a general analytic relation between cumulants and harmonics

$$\kappa_{2m} = \frac{v_n^{2m}\{2m\}}{c_n\{2m\}}. \quad (29)$$

We will follow the notation and definitions of Ref. [16], including the transverse event flow vector  $Q$  represented as a complex number, see Eq. (17) of Ref. [16], and the definition of the cumulants, Eq. (12) and Eq. (25) and relation to the “flow” harmonics, Eq. (28) of Ref. [16]. The latter equation is derived from the formalism of generating functions and defines the connection between the cumulants  $\langle\langle |Q|^{2k} \rangle\rangle$  and harmonics  $\langle Q \rangle$ . For large multiplicity,

$$\sum \frac{x^{2k}}{(k!)^2} \langle\langle |Q|^{2k} \rangle\rangle = \ln I_0(2x\langle Q \rangle). \quad (30)$$

Expanding the generating equation (30) up to order  $x^{2k}$  and equating the coefficients of  $x^{2k}$  one obtains a relation between the cumulants and harmonics.

Hence the problem reduces to finding the expansion of  $\ln(I_0(2y))$  at  $y = x\langle Q \rangle = 0$ . The Bessel function can be represented as an infinite product

$$I_0(2y) = \prod_{k=1}^{\infty} \left( 1 + \left( \frac{2y}{j_{0,k}} \right)^2 \right) \quad (31)$$

and thus

$$\ln(I_0(2y)) = \sum_{k=1}^{\infty} \ln \left( 1 + \left( \frac{2y}{j_{0,k}} \right)^2 \right) = \sum_{i=0}^{\infty} a_i y^{2i}, \quad (32)$$

where

$$a_i = \frac{(-1)^{i+1}}{i} \sum_{k=1}^{\infty} \left( \frac{2}{j_{0,k}} \right)^{2i}. \quad (33)$$

Order, $2m$	6	8	10	12	14
$1/\kappa_{2m}$	-4	33	-456	9460	-274800
$1/\kappa_{2m}$ from Eq. (35)	-3.97	32.96	-455.886	9459.56	-274797.56

TABLE I: First few non-trivial coefficient  $\kappa_{2m}$  and their approximation.  $\kappa_2 = \kappa_4 = 1$ .

Using the last equation we obtain the wanted relation

$$\kappa_{2m} = \frac{1}{a_m} = (-1)^{m+1} \left[ m!(m-1)! \sum_{k=1}^{\infty} \left( \frac{2}{j_{0,k}} \right)^{2m} \right]^{-1}. \quad (34)$$

The sum  $\sum_{k=1}^{\infty} \left( \frac{2}{j_{0,k}} \right)^{2m}$  can be well approximated by the first term for  $m > 2$  (see the Table I). At large  $m$  this becomes a robust limit:

$$\lim_{m \rightarrow \infty} \kappa_{2m} = (-1)^{m+1} \frac{(j_{0,k}/2)^{2m}}{m!(m-1)!}. \quad (35)$$

In Table I, we list first few coefficients  $\kappa_{2m}$  and their approximation by Eq. (35).

Since the zeros of the Bessel function,  $j_{0,k}$ , are not always convenient, we also derived an alternative expression by noticing that the function  $f(x) = \ln(I_0(2x))$  satisfies the following differential equation:

$$f'' + (f')^2 + \frac{f'}{x} - 4 = 0; \quad f(0) = 0. \quad (36)$$

Expanding  $f = \sum_n a_n x^{2n}$  and equating the coefficient of the same power of  $x$ , one can show that

$$a_i = \frac{(-1)^{i+1}}{2i} \beta_{2i}, \quad (37)$$

where  $\beta_n$  satisfies the following recursive relation

$$\beta_n = \frac{1}{n} \sum_{i=2}^{n-2} \beta_i \beta_{n-i}; \quad \beta_{2i+1} = 0; \quad \beta_2 = 2. \quad (38)$$

The coefficients relating the cumulants and harmonics in terms of  $\beta_{2n}$  are then

$$\kappa_{2n} = (-1)^{n+1} \frac{2}{n!(n-1)!\beta_{2n}}. \quad (39)$$

### Acknowledgments

I am grateful to my collaborators, Adrian Dumitru and Larry McLerran, whose input to the manuscript was invaluable. I am especially thankful to Adrian Dumitru for encouraging me to write this article. I thank Mateusz Ploskon and the organizers of the conference Initial Stages in High-Energy Nuclear Collision 2014 for creating an stimulating environment which motivated me to finish this manuscript. I also thank Robert Pisarski for valuable comments and careful reading of the manuscript.

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